Sensitivity Analysis Methods for Coupled Acoustic-Structural Systems Part I: Modal Sensitivities

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An analytical method is proposed for calculating the sensitivities of the eigenvalues and eigenvectors of coupled acoustic-structural systems for the purpose of reducing vehicle interior noise. The concept of the left and right eigenvectors with regard to the eigenvalue problems of coupled systems is presented first, and four propositions concerning this concept are then proved. Based on these propositions, the formulations for the eigenvalue and eigenvector sensitivity analysis of coupled systems are obtained. It is shown that the new method provides better convergence than the "modal method" and "modified modal method" (extensions of these methods to coupled systems), and it achieves better calculation efficiency than the extended Nelson's method when more than one eigenvector is being considered. The theoretical results and computation equations are verified through application of the method to a coupled acoustic-box structure and an actual vehicle interior noise problem.

Nomenclature

b_j $= (\lambda_j' M - K' + \lambda_j M')\phi_j$
$\bar{b}_j$ $= (\lambda_j' M - K' + \lambda_j M')^T \bar{\phi}_j$
C_a = coordinate conversion matrix of acoustic system
C_s = coordinate conversion matrix of structure system
C_{ij} = see Eqs. (23) and (31)
C_{ij}^* = see Eqs. (47) and (48)
D_{ij} = see Eqs. (28) and (31)
E_{ij} $= \bar{\phi}_i^T (K' - \lambda_j M')\phi_j$
$\bar{E}_{ij}$ $= \bar{\phi}_j^T (K' - \lambda_j M')\phi_i$
e_j = relative error between $\phi_j'^*$ and ϕ_j'
f = excitation force vector of coupled system
I = unit matrix
K = stiffness matrix of coupled system
K_{aa} = stiffness matrix of sound field
K_{ss} = stiffness matrix of structure
K_{sa} = matrix of coupling term
$\tilde{K}_{aa}'$ $= \Phi_a^T K_{aa}' \Phi_a = \Psi_a^T k_a' \Psi_a$
$\tilde{K}_{ss}'$ $= \Phi_s^T K_{ss}' \Phi_s = \Psi_s^T k_s' \Psi_s$
$\tilde{K}_{sa}'$ $= \Phi_s^T K_{sa}' \Phi_a = \Psi_s^T k_{sa}' \Psi_a$
k_a' = derivative of K_{aa} in modal coordinates
k_s' = derivative of K_{ss} in modal coordinates
k_{sa}' = derivative of K_{sa} in modal coordinates
M = mass matrix of coupled system
M_{aa} = mass matrix of sound field
M_{ss} = mass matrix of structure
M_{as} $= -K_{sa}^T$
M_j = see Eq. (46)
$\tilde{M}_{aa}'$ $= \Phi_a^T M_{aa}' \Phi_a = \Psi_a^T m_a' \Psi_a$
$\tilde{M}_{aa}'$ $= \Phi_s^T M_{ss}' \Phi_s = \Psi_s^T m_s' \Psi_s$
$\tilde{M}_{as}'$ $= \Phi_a^T M_{as}' \Phi_s = \Psi_a^T m_{as}' \Psi_s$
m_a' = derivative of M_{aa} in modal coordinates
m_s' = derivative of M_{ss} in modal coordinates
m_{as}' = derivative of M_{as} in modal coordinates
N = number of all nodal coordinates of coupled system
n = number of modes used in eigenvector sensitivity calculation
S_n^m = see Eq. (47)
u = nodal coordinate vector of coupled system
u_s = nodal displacement vector of structure
u_a = sound pressure vector of interior sound field
X_j = solution of Eq. (21)
$\bar{X}_j$ = solution of Eq. (26)
$\alpha_k(\alpha)$ = kth design variable
$\Delta\alpha$ = increment of α
δ = allowable margin of error
ϵ_j $= \bar{\phi}_j^T M' \phi_j$
κ = an arbitrarily chosen constant
Λ = eigenvalue matrix of coupled system
λ = eigenvalue of coupled system
λ_j = jth eigenvalue of coupled system
μ = a given shift
ξ $= \{\xi_s^T, \xi_a^T\}^T$
ξ_a = modal coordinates of acoustic system
ξ_s = modal coordinates of structural system
Φ_a = component of eigenvector matrix relative to u_a
Φ_s = component of eigenvector matrix relative to u_s
ϕ = eigenvector of coupled system
ϕ_j = jth eigenvector of coupled system
ϕ_{aj} = component of ϕ_j relative to u_a
ϕ_{sj} = component of ϕ_j relative to u_s
$\bar{\phi}$ = left eigenvector of coupled system
$\bar{\phi}_j$ = jth left eigenvector of coupled system
$\phi_j'^*$ = sensitivity of ϕ_j calculated by all N eigenvectors
Ψ $= [\Psi_s^T, \Psi_a^T]^T$, eigenvector matrix corresponding to ξ
Ψ_a = component of Ψ relative to ξ_a
Ψ_s = component of Ψ relative to ξ_s
$(\,)'$ = partial derivative of ( ) with respect to design variable α_κ
$\|x\|$ = norm of vector x

Received Feb. 5, 1990; revision received July 12, 1990; accepted for publication Sept. 1, 1990.

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Introduction

AUTOMOBILE manufacturers are faced with a strong requirement today for developing high-quality vehicles in a short period of time. Since interior noise has a strong effect on vehicle salability, it is particularly important to be able to estimate noise levels accurately by means of simulation at the design stage. Such needs have prompted extensive research into simulation techniques, which have now made it possible to predict interior noise levels to a certain extent using finite element and boundary element models.[1-3] However, what is actually required at the design stage is to be able to find suitable structures that will reduce the interior noise level to the target value. Looking at examples of structural improvements made to date, we see that this task has been accomplished so far through a time-consuming and tedious process of repeatedly analyzing partial structural modifications until the desired structure has been found. To overcome this problem, sensitivity analysis was applied for the first time to a vehicle interior noise problem.[4] It was shown that the use of sensitivity analysis made it easy to determine how the analytical model should be modified or the vehicle structure optimized in the case of relatively low frequencies. However, the object treated in that study was a so-called uncoupled structural-acoustic problem. In an uncoupled problem, it is assumed that structural vibration causes noise, but is not influenced by it. The present work, by contrast, focused on a coupled acoustic-structural system, which was treated with the asymmetrical coefficient matrices of the coupled system.

In general, asymmetrical coefficient matrices result in a complex eigenvalue problem. Also, the conventional orthogonality conditions obtained from the system with the symmetrical coefficient matrices will be not tenable. That makes it necessary to use the concept of left and right eigenvectors. In this paper, it is shown that even though the coefficient matrices are asymmetrical, all the eigenvalues and left and right eigenvectors of the coupled system remain real numbers, provided the damping terms are omitted (proposition 1). More importantly, in this paper it is demonstrated that the left eigenvectors of the coupled system can be found directly using the right eigenvectors (proposition 2). This means there is no need to solve the left eigenproblem, and all the computations can be performed with only the right eigenvectors. The orthogonality condition (proposition 3) and normalization condition (proposition 4), which are related to the right eigenvectors, are then shown. These propositions have been used as the basis for deriving formulations for calculating the modal sensitivity coefficients of the coupled system.

Various methods have been presented for calculating the sensitivity of uncoupled structural systems. As reviewed in Ref. 5 by Sutter and others, the major methods are the "modal method,"[6] Nelson's method,[7] and the "modified modal method."[8] Using the foregoing propositions, these methods can be extended to a coupled system. In this paper, a new method is considered. It is shown that this new method has good computational accuracy and provides much better convergence than the extended modal method and modified modal method. Furthermore, it achieves better calculation efficiency than the extended Nelson's method when more than one eigenvector is being considered. Moreover, this method is a general method and can be reduced to the aforementioned methods by choosing a certain value as the shift parameter. This relationship between the new method and the previous methods is expounded, and the validity of the new method is verified on the basis of its calculation performance.

The theoretical results and computation formulations have all been verified through application of the method to a coupled acoustic-box structure and an actual vehicle interior noise problem. The development of this method should lead to significant progress in research into the identification of analytical models and structural optimization with respect to acoustic-structural coupling.

Eigenvalue Problem of Acoustic-Structural Systems

Using a finite element method (FEM) to discretize the basic equations yields the following FEM equation for the coupled acoustic-structural system (see, for example, Refs. 2 and 3):

$$M\ddot{u} + Ku = f \tag{1}$$

where f represents the excitation vector on the coupled system,

$$u = \begin{Bmatrix} u_s \\ u_a \end{Bmatrix}, \qquad K = \begin{bmatrix} K_{ss} & K_{sa} \\ 0 & K_{aa} \end{bmatrix}, \qquad M = \begin{bmatrix} M_{ss} & 0 \\ M_{as} & M_{aa} \end{bmatrix} \tag{2}$$

Also, u_s is the nodal displacement vector of the structure, u_a the sound pressure vector of the interior sound field, M_{ss} and K_{ss} the mass and stiffness matrices of the structure, M_{aa} and K_{aa} the mass and stiffness matrices of the sound field and M_{as} and K_{sa} the coupling term matrices. Here,

$$K_{sa} = -M_{as}^T \tag{3}$$

The following eigenvalue problem is obtained from Eq. (1):

$$(K - \lambda M)\phi = 0 \tag{4}$$

where λ is the eigenvalue and ϕ is the corresponding right eigenvector. From Eq. (4), N eigenmodes, λ_j and ϕ_j ($j = 1,2, \ldots, N$), can be obtained, where N is the total number of nodal coordinates of the coupled system. For the sake of simplicity, it is assumed here that Eq. (4) does not have repeated eigenvalues. Since the coefficient matrices, K and M are not symmetrical, the following orthogonality conditions generally do not hold true for coupled systems:

$$\phi_i^T K \phi_j = 0 \qquad \text{and} \qquad \phi_i^T M \phi_j = 0 \qquad (\text{for } i \neq j) \tag{5}$$

This means that a conventional modal analysis method (i.e., the mode superposition method) cannot be used with only the right eigenvectors ϕ_j. Therefore, we consider the following left eigenvalue problem:

$$\bar{\phi}^T(K - \lambda M) = 0 \tag{6}$$

where $\bar{\phi}$ is called the left eigenvector. Then the following four propositions have been derived for the purpose of conducting a coupled acoustic-structural analysis. (The proofs are given in the appendix.)

Proposition 1

All the eigenvalues and the right and left eigenvectors of Eq. (1) are always real numbers.

Proposition 2

The left eigenvector $\bar{\phi}_i$ of Eq. (6) can be found using the right eigenvector ϕ_i of Eq. (4). That is, when eigenvalue λ_i is not zero,

$$\bar{\phi}_i = \kappa \begin{Bmatrix} \phi_{si} \\ (1/\lambda_i)\phi_{ai} \end{Bmatrix} \tag{7}$$

When eigenvalue λ_i is zero,

$$\bar{\phi}_i = \kappa \begin{Bmatrix} 0 \\ \phi_{ai} \end{Bmatrix} \qquad (\text{for } \phi_{ai} \neq 0) \tag{8}$$

$$\bar{\phi}_i = \kappa \begin{bmatrix} I \\ K_{aa}^{-1} M_{as} \end{bmatrix} \{\phi_{si}\} \qquad (\text{for } \phi_{ai} = 0) \tag{9}$$

Here, ϕ_{si} and ϕ_{ai} are the components of the right eigenvector ϕ_i relative to u_s and u_a, respectively, i.e., $\phi_i = \{\phi_{si}^T, \phi_{ai}^T\}^T$. I is a unit matrix and κ is an arbitrarily chosen constant. For the sake of simplicity, it is assumed in the following discussion that $\kappa = 1$.

Proposition 3

The orthogonality conditions for the coupled systems are as follows:

When the eigenvalue λ_i is not zero,

$$\phi_{si}^T K_{ss} \phi_{sj} + \phi_{si}^T K_{sa} \phi_{aj} + (1/\lambda_i)\phi_{ai}^T K_{aa} \phi_{aj} = 0$$

$$\phi_{si}^T M_{ss} \phi_{sj} + (1/\lambda_i)(\phi_{ai}^T M_{as} \phi_{sj} + \phi_{ai}^T M_{aa} \phi_{aj}) = 0 \qquad (\text{for } i \neq j) \tag{10}$$

When the eigenvalue λ_i is zero,

$$\phi_{ai}^T K_{aa} \phi_{aj} = 0$$

$$\phi_{ai}^T M_{as} \phi_{sj} + \phi_{ai}^T M_{aa} \phi_{aj} = 0 \quad (\text{for } \phi_{ai} \neq 0 \text{ and } i \neq j) \quad (11)$$

$$\phi_{si}^T K_{ss} \phi_{sj} = 0 \qquad \phi_{si}^T (M_{ss} + M_{as}^T K_{aa}^{-1} M_{as}) \phi_{sj}^T$$

$$+ \phi_{si}^T M_{as}^T K_{aa}^{-1} M_{aa} \phi_{aj} = 0 \quad (\text{for } \phi_{ai} = 0 \text{ and } i \neq j) \quad (12)$$

Proposition 4

The M-normalization conditions for the right eigenvectors are as follows:

When the eigenvalue λ_i is not zero,

$$\phi_{si}^T M_{ss} \phi_{si} + (1/\lambda_i)(\phi_{ai}^T M_{as} \phi_{si} + \phi_{ai}^T M_{aa} \phi_{ai}) = 1 \quad (13)$$

When the eigenvalue λ_i is zero,

$$\phi_{ai}^T (M_{aa} + M_{as} K_{ss}^{-1} M_{as}^T) \phi_{ai} = 1 \qquad (\text{for } \phi_{ai} \neq 0) \quad (14)$$

$$\phi_{si}^T (M_{ss} + M_{as}^T K_{aa}^{-1} M_{as}) \phi_{si} = 1 \qquad (\text{for } \phi_{ai} = 0) \quad (15)$$

The following sections will present an investigation of a method for calculating the modal sensitivities of coupled systems based on these four propositions.

Modal Sensitivities of the Coupled Acoustic-Structural Systems

The object of a modal sensitivity analysis is to determine the derivatives (sensitivities) of eigenvalues and eigenvectors relative to the given design variables. Letting $\alpha_k, (k = 1,2, \ldots)$ represent the design variables, the sensitivities of the eigenvalue λ_j, right eigenvector ϕ_j and left eigenvector $\bar{\phi}_j$ in relation to α_k can be written as $\lambda_j'^{\,k}$, $\phi_j'^{\,k}$, and $\bar{\phi}_j'^{\,k}$, respectively. For the sake of simplicity, the superscript k will be omitted in following descriptions.

To begin with, we will consider the right eigenvalue problem in Eq. (4) and let K' and M' represent the derivatives of coefficient matrices K and M relative to the design variable, α.

Performing a partial derivative operation relative to the design variable α on Eq. (4) produces

$$-\lambda_j' M \phi_j + (K - \lambda_j M)\phi_j' = -(K' - \lambda_j M')\phi_j \quad (16)$$

And performing the partial differential operation on M-normalization condition, we can obtain

$$\bar{\phi}_j'^{\,T} M \phi_j + \bar{\phi}_j^T M \phi_j' = -\bar{\phi}_j^T M' \phi_j \quad (17)$$

Premultiplying Eq. (16) by the transposition of the left eigenvector $\bar{\phi}_j$ and using Eq. (6), the derivative of eigenvalue, λ_j', can be obtained as

$$\lambda_j' = \bar{\phi}_j^T (K' - \lambda_j M')\phi_j = E_{jj} \quad (18)$$

To calculate the derivative of right eigenvector ϕ_j', Eq. (16) is rewritten as

$$(K - \lambda_j M)\phi_j' = b_j \quad (19)$$

where

$$b_j = (\lambda_j' M - K' + \lambda_j M')\phi_j \quad (20)$$

A direct solution of Eq. (19) is not possible since $(K - \lambda_j M)$ is singular. To overcome this complication, various methods presented previously for uncoupled structural systems (e.g., Refs. 6–8) could be used for reference, but here a new method will be considered. Let X_j be the solution of

$$(K - \mu M) X_j = b_j \quad (21)$$

where μ is a given shift value. Then, the derivatives of the right eigenvector can be expressed as

$$\phi_j' = X_j + \sum_{i=1}^{n} \phi_i C_{ij} \quad (22)$$

where $n \leq N$. By substituting Eq. (22) into Eq. (19) and applying proposition 3, the coefficients C_{ij} can be obtained as

$$C_{ij} = \frac{\lambda_j - \mu}{\lambda_i - \mu} \frac{1}{\lambda_j - \lambda_i} E_{ij} \qquad (\text{for } i \neq j) \quad (23)$$

where

$$E_{ij} = \bar{\phi}_i^T (K' - \lambda_j M')\phi_j \quad (24)$$

Similarly, the following expression is obtained for the left eigenvector:

$$\bar{\phi}_j' = \bar{X}_j + \sum_{i=1}^{n} \bar{\phi}_i D_{ij} \quad (25)$$

where $\bar{X}_j$ is the solution of

$$(K - \mu M)^T \bar{X}_j = \bar{b}_j \quad (26)$$

and

$$\bar{b}_j = (\lambda_j' M - K' + \lambda_j M')^T \bar{\phi}_j \quad (27)$$

and coefficients D_{ij} are calculated using

$$D_{ij} = \frac{\lambda_j - \mu}{\lambda_i - \mu} \frac{1}{\lambda_j - \lambda_i} \bar{E}_{ij} \qquad (\text{for } i \neq j) \quad (28)$$

where

$$\bar{E}_{ij} = \bar{\phi}_j^T (K' - \lambda_j M')\phi_i \quad (29)$$

The coefficients C_{jj} and D_{jj} in Eqs. (22) and (25) cannot be found from an expression like Eq. (19). Consequently, the values of these coefficients are determined by applying the derivative of normalization conditions, i.e., Eq. (17). Substituting Eqs. (22) and (25) into Eq. (17), the following expression can be obtained:

$$C_{jj} + D_{jj} = -\bar{\phi}_j^T M' \phi_j - \bar{X}_j^T M \phi_j - \bar{\phi}_j^T M X_j \quad (30)$$

By applying proposition 2, i.e., that the component $\bar{\phi}_{sj}$ (or $\bar{\phi}_{aj}$) of the left eigenvector is equal to the component ϕ_{sj} ($\bar{\phi}_{aj}$) of the right eigenvector, the following expression is obtained:

$$C_{jj} = D_{jj} = -\tfrac{1}{2}\epsilon_j \quad (31)$$

where

$$\epsilon_j = \bar{\phi}_j^T M' \phi_j + \bar{X}_j^T M \phi_j + \bar{\phi}_j^T M X_j \quad (32)$$

In general (except $\mu = \lambda_j$), we can prove

$$\bar{X}_j^T M \phi_j = 0 \quad \text{and} \quad \bar{\phi}_j^T M X_j = 0 \quad (33)$$

then ϵ_j can be obtained as

$$\epsilon_j = \bar{\phi}_j^T M' \phi_j \quad (34)$$

It should be noted that the use of proposition 2 makes it possible to carry out the foregoing calculations using only the right eigenvectors $\phi_j (j = 1,2, \ldots, n)$. For the sake of simplicity, it is assumed that there are no eigenvalues having a value of zero. Hence,

$$E_{ij} = \phi_{si}^T (K_{ss}' - \lambda_j M_{ss}')\phi_{sj} + (1/\lambda_i)\phi_{ai}^T (K_{aa}' - \lambda_j M_{aa}')\phi_{aj}$$

$$+ \phi_{si}^T K_{sa}' \phi_{aj} - (\lambda_j/\lambda_i)\phi_{ai}^T M_{as}' \phi_{sj} \quad (35)$$

$$\bar{E}_{ij} = \phi_{sj}^T(K'_{ss} - \lambda_j M'_{ss})\phi_{si} + (1/\lambda_j)\phi_{aj}^T(K'_{aa} - \lambda_j M'_{aa})\phi_{ai} + \phi_{sj}^T K'_{sa}\phi_{ai} - \phi_{aj}^T M'_{as}\phi_{si} \tag{36}$$

$$\epsilon_j = \phi_{sj}^T M'_{ss}\phi_{sj} + (1/\lambda_j)(\phi_{aj}^T M'_{as}\phi_{sj} + \phi_{aj}^T M'_{aa}\phi_{aj}) \tag{37}$$

Most of the approaches employed in reducing interior noise involve modifying the stiffness or mass distribution of the structure. In this case, the design variable α pertains to the structure and only K'_{ss} and M'_{ss} do not have values of zero. Consequently, Eqs. (35–37) can be simplified as noted in the following:

$$E_{ij} = \phi_{si}^T(K'_{ss} - \lambda_j M'_{ss})\phi_{sj}$$

$$\bar{E}_{ij} = E_{ji}$$

$$\epsilon_j = \phi_{sj}^T M'_{ss}\phi_{sj} \tag{38}$$

The layout of the seat, or other obstacles in the interior sound field, is sometimes used to achieve a sound absorption effect. In addition, the air density, velocity of sound of the interior sound field sometimes vary depending on the atmospheric temperature or other conditions. The effect of such variations on the interior noise level can be investigated by letting α represent the sound field design variable and finding the sensitivities in relation to α. In this case, since only K'_{aa} and M'_{aa} do not become zero, Eqs. (35–37) can be simplified as follows:

$$E_{ij} = (1/\lambda_i)\phi_{ai}^T(K'_{aa} - \lambda_j M'_{aa})\phi_{aj}$$

$$\bar{E}_{ij} = (\lambda_j/\lambda_i)E_{ji}$$

$$\epsilon_j = (1/\lambda_j)\phi_{aj}^T M'_{aa}\phi_{aj} \tag{39}$$

In investigating the effects of sound-absorbing materials applied to body panels, it is necessary to change the parameters governing the coupling condition between the body panel and the sound field. In this case, by letting α represent the design variable pertaining to the coupling condition, and assuming that only K'_{sa} and M'_{as} are not zeros, Eqs. (35–37) can be simplified as shown here:

$$E_{ij} = \phi_{si}^T K'_{sa}\phi_{aj} - (\lambda_j/\lambda_i)\phi_{ai}^T M'_{as}\phi_{sj}$$

$$\bar{E}_{ij} = \phi_{sj}^T K'_{sa}\phi_{ai} - \phi_{aj}^T M'_{as}\phi_{si}$$

$$\epsilon_j = (1/\lambda_j)\phi_{aj}^T M'_{as}\phi_{sj} \tag{40}$$

It is clear that the method presented above is also applicable to uncoupled structural or acoustic systems by merely replacing coefficient matrices M and K in Eq. (1) by mass and stiffness matrices of the structure or related coefficient matrices of the acoustic system. In such cases, the left eigenvectors $\bar{\phi}_j$ are equal to the right eigenvectors ϕ_j, and the former expressions can be simplified.

It should also be noted that by choosing a certain value for the shift μ, the method presented earlier can be reduced to the previous methods presented in Refs. 6, 7, and 8 (extension of those methods to coupled systems). In fact, if we let μ have a negative value that approaches infinity, then the solution of Eq. (21), X_j, becomes zero, and the coefficients C_{ij} in Eq. (23) become

$$C_{ij} = \frac{1}{\lambda_j - \lambda_i}E_{ij} \qquad (\text{for } i \neq j) \tag{41}$$

As a result, the method is reduced to the modal method presented by Fox.[6]

If we let $\mu = \lambda_j$, then from Eq. (23), $C_{ij} (i \neq j)$ become zeros, and Eq. (22) can be rewritten as

$$\phi'_j = X_j + C_{jj}\phi_j \tag{42}$$

As a result, the method is reduced to Nelson's method.[7]

And, if we let $\mu = 0$, then in Eq. (23), C_{ij} become

$$C_{ij} = \frac{\lambda_j}{\lambda_i}\,\frac{1}{\lambda_j - \lambda_i}E_{ij} \qquad (\text{for } i \neq j) \tag{43}$$

and the method is reduced to the modified modal method presented by Wang.[8]

In general, an appropriate shift value is $\mu > 0$ and $\mu \neq \lambda_j$, $(j = 1,2, \ldots n)$. The error estimation presented in the following section shows that a shift value $\lambda_{k-1} < \mu < \lambda_k$ is appropriate for calculating eigenvector sensitivities of the kth order or above. Moreover, as μ approaches λ_k, the eigenvector sensitivity error becomes smaller. A comparison of the calculations made with the new method and the previous methods is shown in the following section.

Error Estimation for Ignoring Higher-Order Modes

The foregoing discussion did not touch on the mode number, n. In an eigenvalue sensitivity analysis, the sensitivity of the ith eigenvalue is dependent only on the ith eigenvector. This means that n can be determined according to the number of the eigenvalues that are the object of the sensitivity analysis. In the case of an eigenvector sensitivity analysis, n in Eq. (22) should be equal to the sum of the nodal coordinates of the structure and sound field, N, in order for the results obtained with theory presented in the previous section to be strictly exact. However, in actuality, this is virtually impossible, and, in fact, unnecessary. The following discussion will examine the effect of ignoring the higher-order modes.

It is seen in Eq. (23) that the absolute value C_{ij} decreases as λ_i becomes greater than λ_j. Accordingly, the effect of higher-order eigenvectors can be ignored when analyzing the sensitivity of lower-order eigenvectors. Letting $\phi'_j{}^*$ be the sensitivity of the jth right eigenvector calculated by all N eigenvectors, and ϕ'_j the sensitivity of the jth right eigenvector calculated by the n lower-order eigenvectors, $j \leq n < N$, then, the relative error between $\phi'_j{}^*$ and ϕ'_j can be written as

$$e_j = \|\phi'_j{}^* - \phi'_j\| / \|\phi'_j{}^*\| \tag{44}$$

where $\|x\|$ is the norm of vector $\boldsymbol{x}$. Substituting Eq. (22) into Eq. (44) yields

$$e_j = \Big\| \sum_{i=n+1}^{N} \frac{\lambda_j - \mu}{\lambda_i - \mu}\,\frac{1}{\lambda_j - \lambda_i} E_{ij}\phi_i \Big\| / \|\phi'_j{}^*\| \leq \Big| \frac{\lambda_j - \mu}{\lambda_{n+1} - \mu} \Big| M_j \tag{45}$$

where

$$M_j = \max\,\{S_n^m;\ m = n+1, \ldots, N\} / \|\phi'_j{}^*\| \tag{46}$$

$$S_n^m = \Big\| \sum_{i=n+1}^{m} \frac{1}{\lambda_j - \lambda_i} E_{ij}\phi_i \Big\| = \Big\| \sum_{i=n+1}^{m} \phi_i C_{ij}^* \Big\| \quad (n+1 \leq m \leq N) \tag{47}$$

Because $\|\phi'_j{}^*\|$ can be obtained as

$$\|\phi'_j{}^*\| = \Big\| \sum_{i=1}^{N} \phi_i C_{ij}^* \Big\| \tag{48}$$

where $C_{ij}^* = 1/(\lambda_j - \lambda_i)E_{ij}$ (for $i \neq j$) and $C_{ii}^* = -\tfrac{1}{2}\epsilon_i$. Therefore, in general, $S_n^m \leq \|\phi'_j{}^*\|$, so $M_j \leq 1$. For the sake of simplicity, it can be assumed that $M_j = 1$. From Eq. (45), it can be shown that as μ approaches λ_j, the error e_j becomes smaller, and that when λ_{n+1} is larger than λ_j, the error e_j is smaller. If we assume that δ is the allowable margin of error, then, from the smallest value of λ_{n+1} that satisfies

$$\lambda_{n+1} \geq \mu + (1/\delta)(\lambda_j - \mu) \tag{49}$$

the number of eigenvectors that should be considered for calculating the jth eigenvector sensitivity, n, can be determined.

Implementation of the Method

The method described earlier has been implemented in a sensitivities analysis system (SAS), which is a general-purpose processor for coupled acoustic-structural analysis, on the base of employing the NASTRAN computer code. The following discussion describes the calculation procedure for SAS and how it has been implemented using the NASTRAN computer code.

Calculation of the matrix $E = (E_{ij})$ is the main computation performed in the modal sensitivity analysis. Rewriting Eq. (35) as a matrix results in

$$E = \tilde{K}'_{ss} - \tilde{M}'_{ss}\Lambda + \Lambda^{-1}(\tilde{K}'_{aa} - \tilde{M}'_{aa}\Lambda) + \tilde{K}'_{sa} - \Lambda^{-1}\tilde{M}'_{as}\Lambda \tag{50}$$

where

$$\tilde{K}'_{ss} = \Phi_s^T K'_{ss}\Phi_s, \quad \tilde{K}'_{aa} = \Phi_a^T K'_{aa}\Phi_a, \quad \tilde{K}'_{sa} = \Phi_s^T K'_{sa}\Phi_a$$
$$\tilde{M}'_{ss} = \Phi_s^T M'_{ss}\Phi_s, \quad \tilde{M}'_{aa} = \Phi_a^T M'_{aa}\Phi_a, \quad \tilde{M}'_{as} = \Phi_a^T M'_{as}\Phi_s \tag{51}$$

$$\Phi_s = [\phi_{s1}, \ldots, \phi_{sn}], \quad \Phi_a = [\phi_{a1}, \ldots, \phi_{an}], \quad \Lambda = \mathrm{diag}\{\lambda_i\} \tag{52}$$

In carrying out an eigenvalue analysis of a coupled system, it is desirable to reduce the amount of computation that must be done. An effective way to do this is to condense the degrees of freedom of that structure and sound field using the component modal synthesis method.[9,10] The simplest way is to convert the degrees of freedom of the structure and sound field from physical coordinate systems to their respective modal coordinate systems.[10] This involves the following type of coordinate conversion:

$$\begin{Bmatrix} u_s \\ u_a \end{Bmatrix} = \begin{bmatrix} C_s & 0 \\ 0 & C_a \end{bmatrix} \begin{Bmatrix} \xi_s \\ \xi_a \end{Bmatrix} \tag{53}$$

Here, C_s and C_a are the coordinate conversion matrices of the structure and acoustic system, respectively, and are formed by lower-order eigenvectors of the structure and acoustic system, respectively. The symbols ξ_s and ξ_a are the corresponding modal coordinates.

The eigenvector matrix corresponding to the modal coordinates, $\xi = \{\xi_s^T, \xi_s^T\}^T$ is given as

$$\Psi^T = [\Psi_s^T, \Psi_a^T] \tag{54}$$

Then, according to Eq. (53)

$$\Phi_s = C_s\Psi_s, \qquad \Phi_a = C_a\Psi_a \tag{55}$$

As a result,

$$\tilde{K}'_{ss} = \Psi_s^T k'_s \Psi_s, \quad \tilde{K}'_{aa} = \Psi_a^T k'_a \Psi_a, \quad \tilde{K}'_{sa} = \Psi_s^T k'_{sa}\Psi_a$$
$$\tilde{M}'_{ss} = \Psi_s^T m'_s \Psi_s, \quad \tilde{M}'_{aa} = \Psi_a^T m'_a \Psi_a, \quad \tilde{M}'_{as} = \Psi_a^T m'_{as}\Psi_s \tag{56}$$

where

$$k'_s = C_s^T K'_{ss} C_s, \quad k'_a = C_a^T K'_{aa} C_a, \quad k'_{sa} = C_s^T K'_{sa} C_a$$
$$m'_s = C_s^T M'_{ss} C_s, \quad m'_a = C_a^T M'_{aa} C_a, \quad m'_{as} = C_a^T M'_{as} C_s \tag{57}$$

The notations k'_s, m'_s, m'_a, k'_{sa}, and m'_{as} represent the derivatives of coefficient matrices in the modal coordinate systems.

By approximating the derivatives $(\,)'$ in terms of the finite differences $\Delta(\,)/\Delta\alpha$, and the values k'_s, m'_s, k'_a, and m'_a can be readily obtained from an uncoupled structural and acoustic analysis using the NASTRAN computer code. Where $\Delta\alpha$ represents the increment of design variable α. The remaining calculations are then performed with SAS. In addition, the amount of computation done for Eq. (56) is usually very little as it depends only on the number of modal coordinates used.

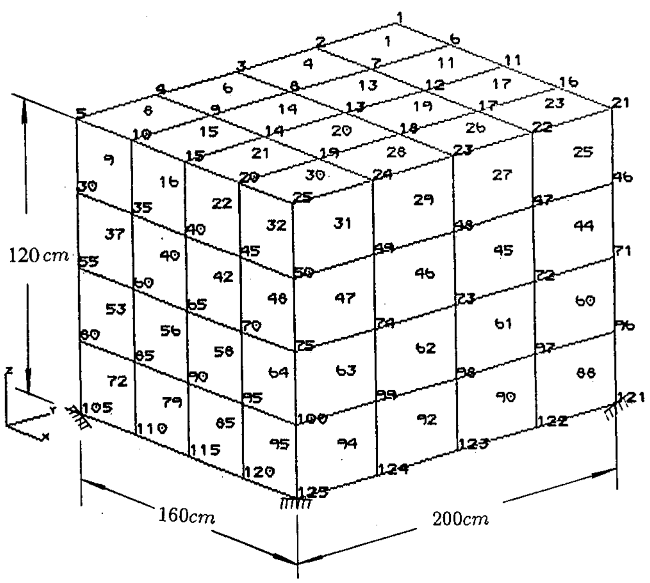


Fig. 1 Model of coupled acoustic-box system.

Table 1 Eigenfrequencies of coupled acoustic-structural box

Mode number	Structural frequency, Hz	Acoustic frequency, Hz	Coupled frequency, Hz
1		1) 0.00	0.00
2	1) 8.60		9.28
3	2) 9.83		9.76
4	3) 13.62		13.56
5	4) 14.71		14.85
6	5) 15.10		15.06
31	30) 83.67		83.25
32		2) 87.19	87.86
33	31) 91.74		92.66
34	32) 96.70		96.29
35	33) 106.1		106.3
36	34) 106.7		106.5
37		3) 109.0	110.3

Table 2 Eigenfrequency sensitivities with respect to design variables $\alpha_1 \sim \alpha_4$

Mode number	α_1	α_2	α_3	α_4
2	2.984	1.664	2.767	3.180
3	4.222	5.215	4.648	0.705
4	6.122	1.643	5.494	4.479
5	5.147	3.885	5.238	4.930
6	4.017	6.018	4.500	5.168
31	10.84	28.75	17.26	36.85
32	6.094	6.068	5.875	6.269
33	29.10	24.30	27.34	19.70
34	23.83	80.27	31.28	24.05
35	53.05	50.92	49.58	25.50
36	15.81	38.85	17.64	30.11
37	5.291	2.000	4.773	5.260

Therefore, it is clear that sensitivity analyses can be carried out at high speed using SAS.

Application of Sensitivity Analysis

Modal Sensitivity Analysis of an Acoustic-Box Model

A model of the coupled acoustic-box system analyzed is shown in Fig. 1. The coupled system was formed inside an empty rectangular parallelepiped made of steel plates and measuring 200 cm in length, 160 cm in width, and 120 cm in height. The box structure had a Young's modulus of 2.1×10^5 Pa, density of 0.8×10^{-6} kg/cm^3 and Poisson ratio of 0.3. The thickness of the steel plates was 0.4 cm.

First, a finite element analysis of the structure and sound field was carried out. The structural model used in the analysis had 98 nodes and 96 quadrilateral plate elements (CQUAD4) and the sound field had 125 nodes and 64 solid elements (CHEXA).[11] Fifty-three eigenmodes was found for the structure and seventeen for the sound field in a frequency domain of 0 to 300 Hz. (The sound field modes included one rigid body mode.) Based on those eigenmodes, the eigenvalues and eigenvectors of the coupled system, Λ and Φ, were found. Table 1 shows the eigenvalues of box structure, sound field, and coupled system, respectively, in frequency domains of 0–20 and 80–110 Hz.

The design variables, α_1, α_2, α_3, α_4, used in the analysis were the respective plate thicknesses of the 1st-10th, 11th-20th, 21st-30th, and 31st-40th shell elements of the model as shown in Fig. 1. The resulting frequency sensitivities are shown in Table 2 for the same frequency domains as in Table 1. (The first mode for the rigid body motion has been omitted.) Where the design variables were changed by 0.001% (i.e., $\Delta\alpha_k/\alpha_k = 0.001\%$, $k = 1, 2, 3, 4$) for calculating the derivatives of coefficient matrices.

Table 3 shows a comparison of the predicted frequencies obtained with the sensitivity results and the exact frequencies obtained by re-analysis. Where the results calculated for a 1% change in the design variable α_1. As the data in the table indicate, a change of 1% in the design variable resulted in approximately a 0.1% change in the eigenfrequencies of the system, and, the eigenfrequencies predicted with this method showed an error of less than 0.01%.

Eigenvector sensitivities tend to vary according to the components of the degree of freedom being considered. In general, the sensitivities of the eigenvector components at excitation (input) points and measurement (output) points are important. In this application, the excitation point was the 40th node along the y axis of the structure and the measurement point was the 32nd node in the sound field. Table 4 gives the sensitivities data calculated for the eigenvector components when $\Delta\alpha_k/\alpha_k = 0.001\% (k = 1, \dots 4)$ and $\mu = 9.27$ Hz. Mode numbers with an asterisk (*) indicate the output point of the sound field and those without an asterisk indicate the input point of the structure.

The effect of higher-order modes on eigenvector sensitivities is shown in Table 5. It was assumed that the sensitivities found for $n = 70$ was the exact value. The sensitivities of the components of the 2nd–4th–, 10th–, and 32nd-order eigenvectors were then calculated for different modal numbers, $n = 34$, 12, 5, and 3. The results in the table indicate that sufficient accuracy can be obtained by using only 1 or 2 higher-order modes than the one under consideration.

The effect on eigenvector sensitivities of different chosen values of shift μ is shown in Table 6. The sensitivities of the 2nd eigenvector components were calculated for different shift values, $\mu = -\infty$, 0.0, 9.20 Hz, and 9.27 Hz, and different modal numbers $n = 3$, 8, 22, 34, and 70. As explained in the previous section, when $\mu = -\infty$, the method is reduced to the modal method,[6] and when $\mu = 0$, the method is reduced to the modified modal method.[7] The results in the table show that the method presented in this paper can obtain a significant improvement in the convergence of the eigenvector sensitivities when μ has a positive value, and that convergence is more rapid when the value of μ is closer to the fundamental elastic eigenfrequency, $f = 9.28$ Hz.

A comparison with Nelson's method[8] in terms of computational efficiency shows that with this method ($\mu \neq \lambda_j$) the

Table 3 Predicted values of eigenfrequencies ($\Delta\alpha_1/\alpha_1 = 1\%$)

Mode number	Original frequencies, Hz	Predicted frequencies, Hz	Exact frequencies, Hz
2	9.281	9.290	9.290
3	9.765	9.779	9.779
4	13.559	13.578	13.578
5	14.848	14.864	14.864
6	15.060	15.073	15.072
31	83.251	83.286	83.286
32	87.864	87.884	87.883
33	92.657	92.751	92.750
34	96.292	96.369	96.368
35	106.26	106.43	106.42
36	106.51	106.56	106.56
37	110.33	110.35	110.35

Table 4 Eigenvector sensitivities with respect to design variables $\alpha_1 \sim \alpha_4$ ($\mu = 9.27$ Hz)

Mode number	α_1	α_2	α_3	α_4
2	4.810E − 1	8.156E − 1	6.069E − 1	5.204E − 1
6	− 2.346E − 1	− 5.791E + 0	7.647E − 1	2.838E + 0
10	1.330E + 0	1.394E − 1	− 1.201E + 0	9.089E − 1
32	− 3.899E + 0	− 3.312E + 0	− 3.679E + 0	− 3.883E + 0
36	− 1.067E + 0	5.194E − 1	− 1.762E + 0	5.875E + 0
2*	9.208E − 7	− 2.848E − 6	4.390E − 7	3.574E − 6
6*	2.129E − 7	1.040E − 5	− 2.919E − 6	− 6.970E − 6
10*	5.249E − 6	− 1.591E − 6	− 5.070E − 6	5.064E − 6
32*	− 1.148E − 4	− 1.073E − 4	− 1.006E − 4	− 9.983E − 5
36*	− 8.356E − 5	− 2.558E − 4	− 1.443E − 4	3.853E − 5

Table 5 Effect of higher-order modes on eigenvector sensitivities ($\Delta\alpha/\alpha = 0.001\%$, $\mu = 9.27$ Hz)

Mode number	Exact	$n = 34$	$n = 12$	$n = 5$	$n = 3$
2	4.810E − 1	4.810E − 1	4.810E − 1	4.810E − 1	4.811E − 1
3	− 1.181E + 0	− 1.181E + 0	− 1.181E + 0	− 1.178E + 0	− 1.198E + 0
4	− 2.610E + 0	− 2.610E + 0	− 2.607E + 0	− 2.660E + 0	——
10	1.330E + 0	1.330E + 0	1.371E + 0	——	——
32	− 3.899E + 0	− 3.825E + 0	——	——	——
2*	9.208E − 7	9.208E − 7	9.208E − 7	9.205E − 7	9.205E − 7
3*	6.699E − 6	6.699E − 6	6.696E − 6	6.751E − 6	6.779E − 6
4*	1.115E − 5	1.115E − 5	1.118E − 5	1.001E − 5	——
10*	− 1.391E − 6	− 1.333E − 6	− 1.937E − 6	——	——
32*	− 1.148E − 4	− 1.141E − 4	——	——	——

inverse matrix of $(K - \mu M)$ in Eq. (21) can be calculated only once. But with Nelson's method, the inverse matrices of $(K - \lambda_j M)$ have to be calculated for each eigenvector ϕ_j. Therefore, the method presented here is more efficient than Nelson's method when more than one eigenvector is being considered.

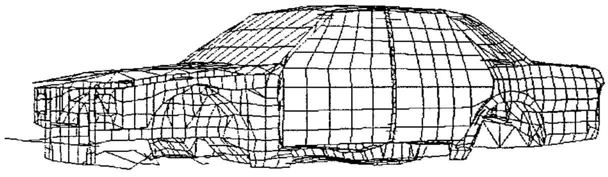
Modal Sensitivity Analysis of Coupled Acoustic-Vehicle Model

To confirm whether the sensitivity analysis method presented here was applicable to large-scale coupled acoustic-structural systems, it was used to conduct calculations for an actual vehicle model (Fig. 2). As shown in the figure, the vehicle model had a total of 2982 nodes and 3713 elements (including 2100 CQUAD4 elements, 694 CTRIA3 elements, 732 CBAR elements, and 187 RBAR elements).[11]

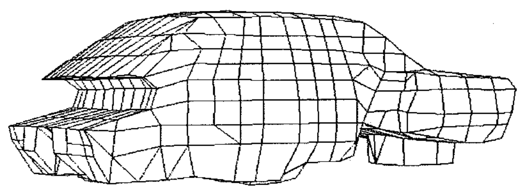
To begin with, the lowest-order mode to the fifth-order mode of the vehicle and sound field were found, respectively, and were used to calculate the eigenmodes of the coupled system. The computation time was 662.7s using a Cray X-MP/EA432 computer. Utilizing the method described here a sensitivity analysis was then performed on the eigenmodes. The design variable was the respective thickness of all CQUAD4 and CTRIA3 elements of the vehicle body. The sensitivity distributions found for the fundamental eigenfrequency and the third-order eigenfrequency are shown in Figs. 3 and 4, respectively. The time required with this method to calculate the sensitivities in relation to 2794 design variables was 1410.0 s. In other words, it took only 0.5 s to calculate the sensitivity in relation to a change in one design variable. The represents a substantial reduction in CPU running time com-

Table 6 Effect of shift μ on eigenvector sensitivity

n	$\mu = -\infty$	$\mu = 0.0$	$\mu = 9.20$Hz	$\mu = 9.27$Hz
3	5.474E − 1	5.054E − 1	4.817E − 1	4.810E − 1
8	4.729E − 1	4.762E − 1	4.810E − 1	4.810E − 1
22	5.117E − 1	4.813E − 1	4.810E − 1	4.810E − 1
34	5.014E − 1	4.811E − 1	4.810E − 1	4.810E − 1
70	4.810E − 1	4.810E − 1	4.810E − 1	4.810E − 1
3*	2.216E − 7	8.037E − 7	9.186E − 7	9.205E − 7
8*	2.293E − 7	7.982E − 7	9.183E − 7	9.205E − 7
22*	9.860E − 7	9.244E − 7	9.209E − 7	9.208E − 7
34*	7.357E − 7	9.206E − 7	9.208E − 7	9.208E − 7
70*	9.208E − 7	9.208E − 7	9.208E − 7	9.208E − 7



a) Model of vehicle body



b) Model of interior sound field

Fig. 2 Model of coupled vehicle body-interior sound field system.

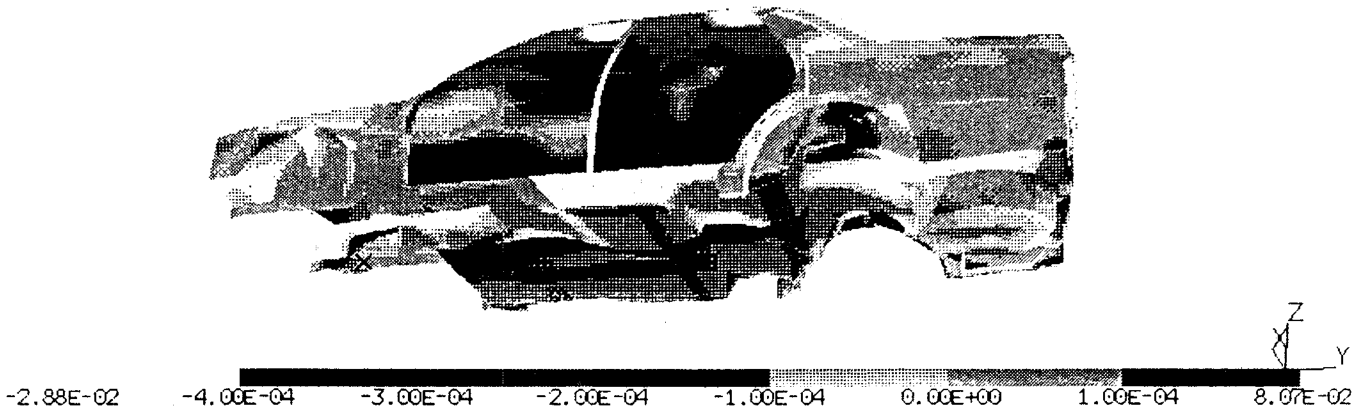


Fig. 3 Sensitivity distribution for fundamental eigenfrequency (f_1 = 13.04 Hz).

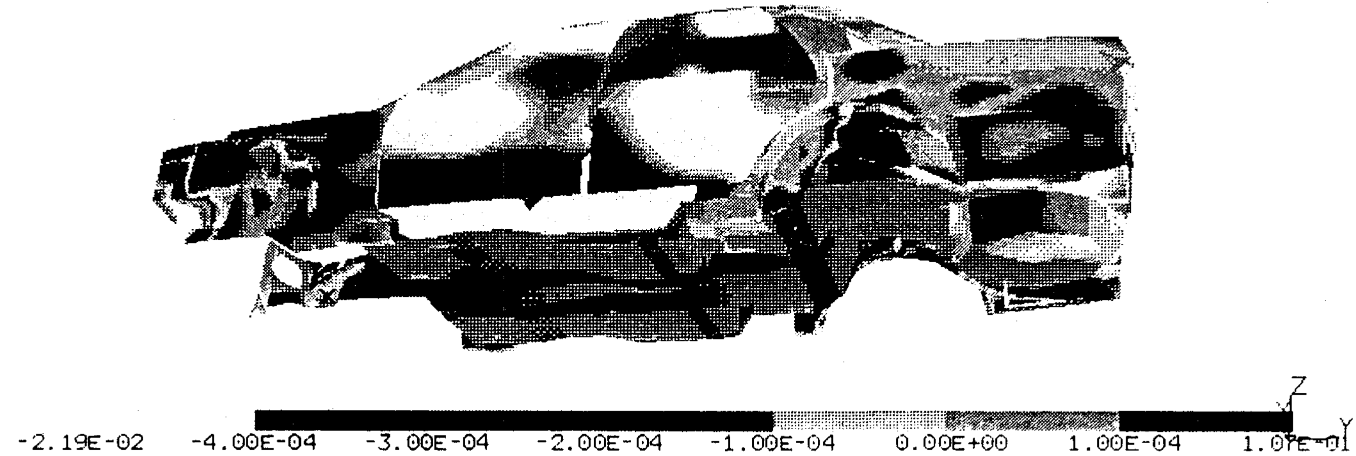


Fig. 4 Sensitivity distribution for 3rd order eigenfrequency (f_3 = 17.07 Hz).

pared with the 662.7 s that are required for re-analyzing the system again. This result confirms that the method proposed here is an effective way to obtain the optimal design for a large-scale coupled acoustic-structural system.

Conclusion

This paper has presented a new modal sensitivity analysis method for analyzing coupled acoustic-structural systems in order to reduce vehicle interior and other internal noise levels. The concept of left and right eigenvectors was first introduced with regard to eigenvalue problems of coupled acoustic-structural systems. Four propositions were then formulated and validated. Based on those propositions, fundamental formulations were devised for conducting a modal sensitivity analysis of a coupled acoustic-structural system. These formulations are applicable to design changes in the structure, sound field, or coupling conditions.

It is shown that the new method has good computational accuracy and provides much better convergence than the extended modal method and modified modal method, and it achieves better calculation efficiency than the extended Nelson's method. Finally, the implementation of the calculation procedure was examined and the sensitivity analysis method was applied to a coupled acoustic-box model and an actual vehicle model. The results verified the effectiveness of this method. In future work, the authors intend to examine the frequency response of coupled acoustic-structural systems along with a sensitivity analysis method for use in this area.

Appendix

Proof of Proposition 1

It will be demonstrated that the arbitrary eigenvalue λ_i in Eq. (4) is a real number. As a result, it will be obvious that the arbitrary eigenvector is also a real number.

Assuming that $\lambda_i = 0$, it is proved that λ_i is a real number. Therefore, we will only consider the case where $\lambda_i \neq 0$. Equation (4) can be expanded to yield the following two expressions:

$$(K_{ss} - \lambda_i M_{ss})\phi_{si} + K_{sa}\phi_{ai} = 0 \qquad \text{(A1)}$$

$$(K_{aa} - \lambda_i M_{aa})\phi_{ai} - \lambda_i M_{as}\phi_{si} = 0 \qquad \text{(A2)}$$

Adopting conjugate complex numbers at both ends of Eqs. (A1) and (A2) results in

$$(K_{ss} - \lambda_i^* M_{ss})\phi_{si}^* + K_{sa}\phi_{ai}^* = 0 \qquad \text{(A3)}$$

$$(K_{aa} - \lambda_i^* M_{aa})\phi_{ai}^* - \lambda_i^* M_{as}\phi_{si}^* = 0 \qquad \text{(A4)}$$

where λ_i^*, ϕ_{si}^*, and ϕ_{ai}^* are the conjugate complex numbers of λ_i, ϕ_{si}, and ϕ_{ai}, respectively.

Premultiplying Eqs. (A1) and (A3) by ϕ_{si}^{*T} and ϕ_{si}^T, respectively, and taking the difference of the two equations yields

$$(\lambda_i^* - \lambda_i)\phi_{si}^{*T}M_{ss}\phi_{si} + \phi_{si}^{*T}K_{sa}\phi_{ai} - \gamma_{si}^T K_{sa}\phi_{ai}^* = 0 \qquad \text{(A5)}$$

Also, premultiplying Eq. (A2) and (A4) by $(1/\lambda_i)\phi_{ai}^{*T}$ and $(1/\lambda_i^*)\phi_{ai}^T$, respectively, and taking the difference of the two equations results in

$$\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_i^*}\right)\phi_{ai}^{*T}K_{aa}\phi_{ai} - \phi_{ai}^{*T}M_{as}\phi_{si} + \phi_{ai}^T M_{as}\phi_{si}^* = 0 \qquad \text{(A6)}$$

Then, by combining Eqs. (A5) and (A6) and using Eq. (3), we obtain

$$(\lambda_i^* - \lambda_i)(\lambda_i\lambda_i^*\phi_{si}^{*T}M_{ss}\phi_{si} + \phi_{ai}^{*T}K_{aa}\phi_{ai}) = 0 \qquad \text{(A7)}$$

So long as λ_i is not zero, it is easy to prove that $(\lambda_i\lambda_i^*\phi_{si}^{*T}M_{ss}\phi_{si} + \phi_{ai}^{*T}K_{aa}\phi_{ai})$ does not become zero. Consequently, from Eq. (A7), λ_i^* must be equal to λ_i. This proves that λ_i is necessarily a real number.

Proof of Proposition 2

By substituting Eq. (7) into Eq. (6) and expanding the expression, we obtain

$$\phi_{si}^T(K_{ss} - \lambda_i M_{ss}) - \phi_{ai}^T M_{as} = 0$$

$$\phi_{ai}^T(K_{aa} - \lambda_i M_{aa}) + \lambda_i\phi_{si}^T K_{sa} = 0 \qquad \text{(A8)}$$

Transposing Eq. (A8) and using the symmetry of the K_{ss}, M_{ss}, K_{aa}, and M_{aa} matrices along with Eq. (3) results in

$$(K_{ss} - \lambda_i M_{ss})\phi_{si} + K_{sa}\phi_{ai} = 0$$

$$(K_{aa} - \lambda_i M_{aa})\phi_{ai} - \lambda_i M_{as}\phi_{si} = 0 \qquad \text{(A9)}$$

Equation (A9) represents the expanded form of Eq. (4), i.e., Eqs. (A1) and (A2), and so Eq. (7) is proved. Equations (8) and (9) can be proved in the same way.

Proof of Proposition 3

a) When the eigenvalue λ_i is not zero

Premultiplying $(K - \lambda_j M)\phi_j = 0$ by $\bar{\phi}_i^T$ and using Eq. (7) results in

$$\begin{aligned}\bar{\phi}_i^T(K - \lambda_j M)\phi_j &= \phi_{si}^T(K_{ss} - \lambda_j M_{ss})\phi_{sj} \\ &+ \phi_{si}^T K_{sa}\phi_{aj} - (\lambda_j/\lambda_i)\phi_{ai}^T M_{as}\phi_{sj} \\ &+ (1/\lambda_i)\phi_{ai}^T(K_{aa} - \lambda_j M_{aa})\phi_{aj} = 0\end{aligned} \qquad \text{(A10)}$$

Postmultiplying $\bar{\phi}_i^T(K - \lambda_i M) = 0$ by ϕ_j and using Eq. (7) yields

$$\begin{aligned}\bar{\phi}_i^T(K - \lambda_i M)\phi_j &= \phi_{si}^T(K_{ss} - \lambda_i M_{ss})\phi_{sj} \\ &+ \phi_{si}^T K_{sa}\phi_{aj} - \phi_{ai}^T M_{as}\phi_{sj} \\ &+ (1/\lambda_i)\phi_{ai}^T(K_{ss} - \lambda_i M_{aa})\phi_{aj} = 0\end{aligned} \qquad \text{(A11)}$$

Subtracting Eq. (A11) from Eq. (A10) results in

$$(\lambda_i - \lambda_j)[\phi_{si}^T M_{ss}\phi_{sj} + (1/\lambda_i)(\phi_{ai}^T M_{as}\phi_{sj} + \phi_{ai}^T M_{aa}\phi_{aj})] = 0 \qquad \text{(A12)}$$

Since $i \neq j$, $\lambda_i - \lambda_j$ is not equal to zero, the following expression must be true:

$$\phi_{si}^T M_{ss}\phi_{sj} + (1/\lambda_i)(\phi_{ai}^T M_{as}\phi_{sj} + \phi_{ai}^T M_{aa}\phi_{aj}) = 0 \quad (\text{for } i \neq j) \qquad \text{(A13)}$$

Substituting Eq. (A13) into Eq. (A11) results in

$$\phi_{si}^T K_{ss}\phi_{sj} + \phi_{si}^T K_{sa}\phi_{aj} + (1/\lambda_i)\phi_{ai}^T K_{aa}\phi_{aj} = 0 \quad (\text{for } i \neq j) \qquad \text{(A14)}$$

b) When the eigenvalue λ_i is equal to zero

Since $\lambda_i = 0$, the following expression is obtained from the left eigenvalue problem

$$\bar{\phi}_i^T K = 0 \qquad \text{(A15)}$$

Premultiplying $(K - \lambda_j M)\phi_j = 0$ by $\bar{\phi}_i^T$ results in

$$\bar{\phi}_i^T K\phi_j = 0 \quad \text{and} \quad \bar{\phi}_i^T M\phi_j = 0 \ (\text{for } i \neq j) \qquad \text{(A16)}$$

By substituting Eqs. (8) and (9) into Eq. (A16), respectively, we obtain Eqs. (11) and (12).

Proof of Proposition 4

By substituting Eqs. (7-9) into the normalization condition,

$$\bar{\phi}_i^T M \phi_i = 1 \tag{A17}$$

respectively, and using $\phi_{si} = -K_{ss}^{-1} K_{sa} \phi_{ai}$, (for $\lambda_i = 0$ and $\phi_{ai} \neq 0$), we obtain Eqs. (13-15).

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